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Vertex-disjoint claws in graphs

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Abstract

Let $\delta(G)$ denote the minimum degree of a graph G. We prove that a graph G of order at least $4k+6$ with $\delta(G) \geqslant k+2$ contains k pairwise vertex-disjoint $K_{1,3}$'s. The conditions on the minimum degree and on the order of the graph are best possible in a sense.

Keywords: Claw; Vertex-disjoint subgraphs; Minimum degree

1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. Let F be a given connected graph. Suppose that $|V(G)|$ is a multiple of $|V(F)|$. A spanning subgraph of G is called an *F-factor* if its components are all isomorphic to F.

There are many results concerning minimum degree conditions for a graph to have an F-factor. Corrádi and Hajnal [2] proved that $\delta(G) \geqslant \frac{2}{3}|V(G)|$ suffices for the existence of a K_3-factor. Enomoto et al. [5] proved for $F = P_3$ that $\delta(G) \geqslant \frac{1}{3}|V(G)|$ is sufficient if G is connected. Hajnal and Szemerédi [7] proved for $F = K_t$ that $\delta(G) \geqslant ((t-1)/t)|V(G)|$ suffices. More generally, Alon and Yuster [1] proved an asymptotic result, which states that $\delta(G) \geqslant (1 - 1/\chi(F))|V(G)|$ assures the existence of an F-factor, where $\chi(F)$ denotes the chromatic number of F.

In the case where F is a *claw*, i.e., $F = K_{1,3}$, Alon and Yuster's result implies that $\delta(G) \geqslant (1/2 + o(1))|V(G)|$ suffices. (Note that claws in this paper are not required to be induced.) In fact, it is proved in [3] that $\delta(G) \geqslant \frac{1}{2}|V(G)|$ is sufficient unless G is isomorphic to $K_{2k,2k}$ with k being odd. However, if we want to find k pairwise vertex-disjoint claws in a graph of order slightly larger than $4k$, a much weaker condition on the minimum degree guarantees the existence. Our main result is the following.

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Main Theorem. *If G is a graph with $|V(G)| \geqslant 4k+6$ and $\delta(G) \geqslant k+2$, then G contains k pairwise vertex-disjoint claws.*

Let G be obtained from a cycle of length N with $N \geqslant 3k+1$ by adding $k-1$ vertices that are adjacent to all vertices of the cycle. It is clear that G does not contain k pairwise vertex-disjoint claws and has minimum degree $k+1$. This shows that the minimum degree condition $\delta(G) \geqslant k+2$ is best possible.

The condition on the order of the graph is also best possible for any integer k with $k=4,7,8$ and $k \geqslant 10$. To see this, let a, b and c be the integers satisfying $a+b+c=k-1$ and $\lfloor \frac{k-1}{3} \rfloor = a \leqslant b \leqslant c = \lceil \frac{k-1}{3} \rceil$, and consider the graph $G = K_{4a+3} \cup K_{4b+3} \cup K_{4c+3}$. It is obvious that G contains at most $a+b+c=k-1$ pairwise vertex-disjoint claws, and the minimum degree is $4a+2 = 4\lfloor \frac{k-1}{3} \rfloor + 2 \geqslant k+2$.

However, if we assume that G is connected, the condition $|V(G)| \geqslant 4k+6$ seems not to be best possible. We conjecture that if G is a connected graph with $|V(G)| \geqslant 4k+3$ and $\delta(G) \geqslant k+2$, then G contains k pairwise vertex-disjoint claws. Furthermore, we have not found any 2-connected graph of order $4k+1$ with $\delta(G)=k+2$ which does not contain k pairwise vertex-disjoint claws.

We need the following notation and terminology. Let G be a graph. We denote by $\omega(G)$ the number of components of G. For a vertex $v \in V(G)$, we denote by $N(v) = N_G(v)$ and $\deg(v) = \deg_G(v)$ the set of vertices adjacent to v and the degree of v, respectively. For a vertex set $S \subset V(G)$, we write $\langle S \rangle = \langle S \rangle_G$ for the subgraph of G induced by S. For disjoint subsets S and T of $V(G)$, we let $E(S,T) = E_G(S,T)$ denote the set of edges of G joining a vertex in S and a vertex in T. When S or T consists of a single vertex, say $S=\{x\}$ or $T=\{y\}$, we write $E(x,T)$ or $E(S,y)$ for $E(S,T)$.

2. Preparation for the proof of the main theorem

The subsequent four sections (including this section) are devoted solely to the proof of the main theorem.

We assume that there exists a graph G with $|V(G)| \geqslant 4k+6$ and $\delta(G) \geqslant k+2$ such that G does not contain k pairwise vertex-disjoint claws. Suppose that G is an edge-maximal counterexample. Then G contains $k-1$ vertex-disjoint claws, say $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$. Let $H = G - (\bigcup_{i=1}^{k-1} V(C^{(i)}))$. We suppose that $C^{(1)}, C^{(2)}, \ldots, C^{(k-1)}$ are chosen so that

(a) $|E(H)|$ is maximum,

(b) subject to (a), $\omega(H)$ is minimum, and

(c) subject to (a) and (b), $\sum_{i=1}^{k-1} |E(\langle V(C^{(i)}) \rangle)|$ is maximum.

By the assumption, H contains no claw, or equivalently, every vertex of H has degree at most two. We define $n = |V(H)|$. Note that $n = |V(G)| - 4(k-1) \geqslant (4k+6) - (4k-4) = 10$.

For each i, let $a^{(i)}$ be the center of $C^{(i)}$ and $B^{(i)} = \{b_1^{(i)}, b_2^{(i)}, b_3^{(i)}\}$ be the set of leaves of $C^{(i)}$. In the following argument, we sometimes fix i and set $C = C^{(i)}$. In such cases, we write a, B, b_1, b_2 and b_3 instead of $a^{(i)}$, $B^{(i)}$, $b_1^{(i)}$, $b_2^{(i)}$ and $b_3^{(i)}$, respectively.

Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be the triangular components of H, i.e., the components of H isomorphic to K_3. Define

$$U = \bigcup_{\alpha=1}^{s} V(P^{(\alpha)}) \quad \text{and} \quad W = V(H) - U.$$

In the rest of this section, we shall settle the case where $s \geqslant 3$.

First consider the case $s \geqslant 4$. For each α with $1 \leqslant \alpha \leqslant 4$, we take a vertex $u_\alpha \in V(P^{(\alpha)})$. Since

$$\sum_{i=1}^{k-1} \sum_{\alpha=1}^{4} |E(V(C^{(i)}), u_\alpha)| = \sum_{\alpha=1}^{4} (\deg_G(u_\alpha) - 2) \geqslant 4k,$$

there exists an index i with $1 \leqslant i \leqslant k-1$ such that $\sum_{\alpha=1}^{4} |E(V(C^{(i)}), u_\alpha)| > 4$. Then there exist two edges xu_α and yu_β joining $V(C^{(i)})$ and $\{u_1, u_2, u_3, u_4\}$ with $x, y \in V(C^{(i)})$, $x \neq y$ and $\alpha \neq \beta$. Replacing $C^{(i)}$ by claws contained in $\langle \{x\} \cup V(P^{(\alpha)}) \rangle$ and $\langle \{y\} \cup V(P^{(\beta)}) \rangle$, we obtain k pairwise vertex-disjoint claws in G. This is a contradiction.

Next consider the case $s = 3$. For each α with $1 \leqslant \alpha \leqslant 3$, we take a vertex $u_\alpha \in V(P^{(\alpha)})$, and take a vertex $v \in W$.

Lemma 2.1. *If $C = C^{(i)}$ satisfies $|E(V(C), \{u_1, u_2, u_3, v\})| \geqslant 5$, then*
(i) $2 \leqslant |E(V(C), v)| \leqslant 3$, *and*
(ii) $E(B, \{u_1, u_2, u_3\}) = \emptyset$.

Proof. If $|E(V(C), v)| = 4$, then since $|E(V(C), \{u_1, u_2, u_3, v\})| \geqslant 5$, there exists an edge xu_α with $x \in V(C)$ and $1 \leqslant \alpha \leqslant 3$. Then both $\langle (V(C) - \{x\}) \cup \{v\} \rangle$ and $\langle \{x\} \cup V(P^{(\alpha)}) \rangle$ contain a claw, which is a contradiction. Hence $|E(V(C), v)| \leqslant 3$.

If $E(V(C), v) = \emptyset$, then we have $|E(V(C), \{u_1, u_2, u_3\})| \geqslant 5$. This implies that there exist two independent edges joining $V(C)$ and $\{u_1, u_2, u_3\}$, and hence $\langle V(C) \cup U \rangle$ contains two vertex-disjoint claws. This is a contradiction. Hence $|E(V(C), v)| \geqslant 1$.

To show (i) and (ii), we first suppose that $av \in E(G)$. Then, any edge $b_p u_\alpha \in E(B, \{u_1, u_2, u_3\})$ would make two vertex-disjoint claws in $\langle (V(C) - \{b_p\}) \cup \{v\} \rangle$ and $\langle \{b_p\} \cup V(P^{(\alpha)}) \rangle$, and hence (ii) follows. By (ii), $E(V(C), \{u_1, u_2, u_3\}) = E(a, \{u_1, u_2, u_3\})$, and hence $|E(a, \{u_1, u_2, u_3\})| \leqslant 3$, implying that $|E(V(C), v)| \geqslant 2$. This shows (i).

We may assume that $av \notin E(G)$. Since $|E(V(C), v)| \geqslant 1$, there exists an edge $b_p v$ with $1 \leqslant p \leqslant 3$. We claim that $E(B - \{b_p\}, \{u_1, u_2, u_3\}) = \emptyset$. Assume that there exists an edge $b_q u_\alpha$ with $q \neq p$. If we replace C by the claw with center u_α contained in $\langle \{b_q\} \cup V(P^{(\alpha)}) \rangle$ and set $H' = \langle (V(H) - V(P^{(\alpha)})) \cup (V(C) - \{b_q\}) \rangle$, then we have $|E(H')| \geqslant |E(H)|$ and $\omega(H') < \omega(H)$. This contradicts the maximality of $|E(H)|$ or the minimality of $\omega(H)$. Hence $E(B - \{b_p\}, \{u_1, u_2, u_3\}) = \emptyset$ holds, as claimed. Now, since $E(V(C), \{u_1, u_2, u_3\}) = E(\{a, b_p\}, \{u_1, u_2, u_3\})$ cannot contain two independent edges, it contains at most three edges. This implies that $|E(V(C),v)|\geqslant 2$, and (i) follows. Now, we have another edge $b_q v$ with $q\neq p$. Applying the previous claim to this edge, we have $E(B-\{b_q\},\{u_1,u_2,u_3\})=\emptyset$, and hence $E(B,\{u_1,u_2,u_3\})=\emptyset$. This completes the proof. □

Define

$$J=\{i\mid 1\leqslant i\leqslant k-1, |E(V(C^{(i)}),\{u_1,u_2,u_3,v\})|\geqslant 5\}.$$

Lemma 2.2. $\sum_{i\in J}|E(V(C^{(i)}),v)|\geqslant |J|+4.$

Proof. Let $|J|=m$. By the definition of J, if $i\notin J$ then $|E(V(C^{(i)}),\{u_1,u_2,u_3,v\})|\leqslant 4$. For $i\in J$, by Lemma 2.1(ii), $|E(V(C^{(i)}),\{u_1,u_2,u_3\})|\leqslant 3$ holds. Since $\delta(G)\geqslant k+2$ and the maximum degree of H is two, we have

$$\begin{aligned}4k &\leqslant \left|E\left(\bigcup_{i=1}^{k-1}V(C^{(i)}),\{u_1,u_2,u_3,v\}\right)\right|\\ &= \sum_{i\notin J}|E(V(C^{(i)}),\{u_1,u_2,u_3,v\})|\\ &\quad+\sum_{i\in J}|E(V(C^{(i)}),\{u_1,u_2,u_3\})|+\sum_{i\in J}|E(V(C^{(i)}),v)|\\ &\leqslant 4(k-1-m)+3m+\sum_{i\in J}|E(V(C^{(i)}),v)|.\end{aligned}$$

Thus $\sum_{i\in J}|E(V(C^{(i)}),v)|\geqslant m+4$ follows. □

We may assume that $J=\{1,2,\ldots,m\}$ where $m=|J|$, and

$$|E(V(C^{(1)}),v)|\geqslant|E(V(C^{(2)}),v)|\geqslant\cdots\geqslant|E(V(C^{(m)}),v)|.$$

Then by Lemmas 2.1(i) and 2.2, one of the following statements holds:

(a) $m\geqslant 2$, $|E(V(C^{(1)}),v)|=3$ and $|E(V(C^{(2)}),v)|\geqslant 2$.

(b) $m\geqslant 3$ and $|E(V(C^{(1)}),v)|=|E(V(C^{(2)}),v)|=|E(V(C^{(3)}),v)|=2$.

Case (a): We can take $X^{(1)}\subset V(C^{(1)})$ and $X^{(2)}\subset V(C^{(2)})$ such that

$$X^{(1)}\subset N(v)\cap V(C^{(1)})-\{a^{(1)}\},\quad |X^{(1)}|=2,$$

$$X^{(2)}\subset N(v)\cap V(C^{(2)})-\{a^{(2)}\},\quad |X^{(2)}|=1.$$

By Lemma 2.1(i) and (ii), it follows that $|N(a^{(i)})\cap\{u_1,u_2,u_3\}|\geqslant 2$ for $i=1,2$. Hence we can take two independent edges $a^{(1)}u_\alpha$ and $a^{(2)}u_\beta$. Then, each of $\langle X^{(1)}\cup X^{(2)}\cup\{v\}\rangle$, $\langle\{a^{(1)}\}\cup V(P^{(\alpha)})\rangle$ and $\langle\{a^{(2)}\}\cup V(P^{(\beta)})\rangle$ contains a claw. These claws together with $C^{(3)},\ldots,C^{(k-1)}$ form k pairwise vertex-disjoint claws in G. This contradicts the assumption that G is a counterexample.

Case (b): In this case, since $3 \leqslant |E(V(C^{(i)}), \{u_1, u_2, u_3\})| = |E(a^{(i)}, \{u_1, u_2, u_3\})|$ for $i = 1, 2, 3$, we have $a^{(i)}u_\alpha \in E(G)$ for all i and α with $1 \leqslant i, \alpha \leqslant 3$. We take a vertex $x^{(i)} \in N(v) \cap V(C^{(i)}) - \{a^{(i)}\}$ for $i = 1, 2, 3$. Then, each of $\langle v, x^{(1)}, x^{(2)}, x^{(3)} \rangle$ and $\langle \{a^{(i)}\} \cup V(P^{(i)}) \rangle$ $(1 \leqslant i \leqslant 3)$ contains a claw, contradicting the assumption.

This complete the proof of the main theorem in the case where $s = 3$. □

3. Basic lemmas

In this section, assuming $s \leqslant 2$, we prove several basic lemmas concerning the number of edges between $V(C^{(i)})$ and $V(H)$, which will be used frequently in the subsequent sections. Throughout this section, we fix i with $1 \leqslant i \leqslant k-1$. So a denotes the center of $C = C^{(i)}$, and $B = \{b_1, b_2, b_3\}$ denotes the set of leaves of C.

Lemma 3.1. *If $v \in V(H)$ is adjacent to the vertex a, then $|E(b_p, V(H) - \{v\})| \leqslant \deg_H(v)$ holds for every $b_p \in B$.*

Proof. Replace C by the claw with center a contained in $\langle (V(C) - \{b_p\}) \cup \{v\} \rangle$. The result follows immediately from the maximality of $|E(H)|$. □

Lemma 3.2. *If $v \in V(H)$ is adjacent to all vertices in B, then $|E(a, V(H) - \{v\})| \leqslant \deg_H(v)$.*

Proof. Replace C by the claw with center v contained in $\langle (V(C) - \{a\}) \cup \{v\} \rangle$. The result follows immediately from the maximality of $|E(H)|$. □

Lemma 3.3. *If a vertex $v \in V(H)$ with $\deg_H(v) = 2$ is adjacent to a vertex in B, then $E(a, V(H) - \{v\} - N_H(v)) = \emptyset$.*

Proof. Suppose that $b_p v \in E(G)$ and $av' \in E(G)$ for $v' \in V(H) - \{v\} - N_H(v)$. Then each of $\langle \{b_p, v\} \cup N_H(v) \rangle$ and $\langle \{a, v'\} \cup (B - \{b_p\}) \rangle$ contains a claw, which implies that G contains k pairwise vertex-disjoint claws, a contradiction. □

Lemma 3.4. *If $E(a, V(H)) \neq \emptyset$ and $|E(b_p, V(H))| \geqslant 3$ for a vertex $b_p \in B$, then $|E(b_p, V(H))| = 3$ and $N(a) \cap V(H) \subset N(b_p) \cap V(H)$.*

Proof. Otherwise, we can find in $\langle V(C) \cup V(H) \rangle$ a claw with center b_p and a claw with center a which are mutually vertex-disjoint, a contradiction. □

Lemma 3.5. *If $|(N(b_p) \cup N(b_q)) \cap V(H)| \geqslant 5$ for b_p, $b_q \in B$ with $p \neq q$, then $|E(b_p, V(H))| \leqslant 1$ or $|E(b_q, V(H))| \leqslant 1$.*

Proof. Otherwise, we can find vertex-disjoint claws with centers b_p and b_q in $\langle V(C) \cup V(H) \rangle$, a contradiction. □

Lemma 3.6. *Let u and w be distinct vertices in W with $\deg_H(u) = \deg_H(w) = 2$. If $N(u) \cup N(w) \supset V(C)$, then $|E(V(C), u)| \leqslant 1$ or $|E(V(C), w)| \leqslant 1$.*

Proof. Suppose that $|E(V(C),u)| \geqslant 2$ and $|E(V(C),w)| \geqslant 2$. Since $N(u) \cup N(w) \supset V(C)$, $V(C)$ can be partitioned into X and Y such that $X \subset N(u)$, $Y \subset N(w)$ and $|X| = |Y| = 2$. On the other hand, since $\deg_H(u) = \deg_H(w) = 2$ and u and w are not contained in a triangular component of H, we can choose distinct vertices $x \in N_H(u) - \{w\}$ and $y \in N_H(w) - \{u\}$. Then $\langle \{u,x\} \cup X \rangle$ contains a claw, $\langle \{w,y\} \cup Y \rangle$ contains a claw, and these claws are vertex-disjoint, a contradiction. □

Lemma 3.7. *Let v be a vertex in W with $\deg_H(v) = 2$, and let $N_H(v) = \{u, w\}$. Suppose that $\deg_H(u) = \deg_H(w) = 2$ and $E(B, v) \neq \emptyset$. Then*

(i) *$|E(B, \{u, w\})| \leqslant 4$, and*

(ii) *if equality holds in (i) and $|E(B, v)| = 3$, then $N(u) \cap B = N(w) \cap B$.*

Proof. We may assume that $|E(B,u)| \leqslant |E(B,w)|$. We set $N_H(u) = \{v,x\}$ and $N_H(w) = \{v,y\}$. Note that $x \neq w$ and $y \neq u$ since $v \in W$, but it is possible that $x = y$.

To show (i), we assume that $|E(B,\{u,w\})| \geqslant 5$. Then, we have $|E(B,w)| = 3$ and $|E(B,u)| \geqslant 2$. We may assume that $b_1u, b_2u \in E(G)$. If $E(\{b_1,b_2\},v) \neq \emptyset$, say $b_1v \in E(G)$ without loss of generality, then $\langle a, b_1, u, v \rangle$ contains a claw with center b_1 and $\langle b_2, b_3, w, y \rangle$ contains a claw with center w, a contradiction. Otherwise, we have $b_3v \in E(G)$ because $E(B,v) \neq \emptyset$. Then, $\langle a, b_3, v, w \rangle$ and $\langle b_1, b_2, u, x \rangle$ contain a claw, a contradiction. Thus it is proved that $|E(B,\{u,w\})| \leqslant 4$.

To show (ii), we suppose that $|E(B,\{u,w\})| = 4$ and $|E(B,v)| = 3$. If $N(u) \cap B \neq N(w) \cap B$, then we can find a vertex $b_p \in B$ such that b_p is adjacent to u and the rest of the vertices in B are adjacent to w. Then, $\langle a, b_p, u, v \rangle$ contains a claw with center b_p and $\langle (B - \{b_p\}) \cup \{w, y\} \rangle$ contains a claw with center w. This is a contradiction. □

Lemma 3.8. *Suppose that $\{u, v, w, x\} \subset W$ induces a path component in H such that uv, vw and $wx \in E(H)$. Then*

(i) *$|E(B, \{u, w\})| \leqslant 4$, and*

(ii) *if equality holds in* (i), *then one of the followings holds:* (a) *$N(u) \cap B = N(w) \cap B$; or* (b) *$|N(u) \cap B| = 1$ and $|N(w) \cap B| = 3$.*

Proof. If there exists a vertex $b_p \in B$ such that $b_pw \in E(G)$ and $|E(B - \{b_p\}, u)| = 2$, then by replacing C by a claw contained in $\langle b_p, v, w, x \rangle$, we get a contradiction to the maximality of $|E(H)|$. This implies that if $|E(B,u)| = 3$ then $E(B,w) = \emptyset$, and if $|E(B,u)| = 2$ then $N(w) \cap B \subset N(u) \cap B$. This proves the lemma. □

Lemma 3.9. *Let P be a triangular component of H. If $E(V(C), V(H) - V(P)) \neq \emptyset$, then $|E(V(C), V(P))| \leqslant 3$.*

Proof. If $E(a, V(H) - V(P)) \neq \emptyset$, then by Lemma 3.3, $E(B,u) = \emptyset$ for every vertex $u \in V(P)$, and hence $|E(V(C), V(P))| = |E(a, V(P))| \leqslant 3$ follows. Thus we may assume that $E(a, V(H) - V(P)) = \emptyset$. Without loss of generality, we may assume that $E(b_1, V(H) - V(P)) \neq \emptyset$. If $E(b_p, V(P)) \neq \emptyset$ for some $p \neq 1$, then by replacing C by a claw contained in $\langle \{b_p\} \cup V(P) \rangle$, we get a contradiction to the maximality of $|E(H)|$ or the minimality of $\omega(H)$. Thus $E(\{b_2, b_3\}, V(P)) = \emptyset$. Suppose that $|E(V(C), V(P))| = |E(\{a, b_1\}, V(P))| \geqslant 4$. Then there exist two independent edges ax and $b_1 y$ with x, $y \in V(P)$. By replacing C by a claw contained in $\langle a, b_2, b_3, x \rangle$, we get a contradiction to the maximality of $|E(H)|$ or the minimality of $\omega(H)$. □

Lemma 3.10. *Let P be a triangular component of H. If there exists a vertex $v \in V(H) - V(P)$ such that $|E(V(C), v)| \geqslant 2$, then $E(B, V(P)) = \emptyset$, and hence it follows that $|E(V(C), V(P))| \leqslant 3$.*

Proof. If $b_p u \in E(G)$ for some $b_p \in B$ and $u \in V(P)$, then since $E(V(C) - \{b_p\}, v) \neq \emptyset$, by replacing C by a claw contained in $\langle \{b_p\} \cup V(P) \rangle$, we get a contradiction to the maximality of $|E(H)|$ or the minimality of $\omega(H)$. □

Lemma 3.11. *Let P be a triangular component of H. If there exists a vertex $v \in V(H) - V(P)$ such that $\deg_H(v) + |E(V(C), v)| \geqslant 4$, then $E(V(C), V(P)) = \emptyset$.*

Proof. Assume that $E(x, V(P)) \neq \emptyset$ for some $x \in V(C)$. Then $\langle \{x\} \cup V(P) \rangle$ contains a claw. Also, since $\deg_H(v) + |E(V(C) - \{x\}, v)| \geqslant 3$, $\langle \{v\} \cup N_H(v) \cup (V(C) - \{x\}) \rangle$ contains a claw with center v. This is a contradiction. □

Lemma 3.12. *Suppose that $b_p \in B$ is adjacent to two vertices u and w in H. Then, the following inequality holds*:

$$|E(a, V(H))| + |E(b_p, V(H))| + \deg_H(u) + |E(V(C), u)|$$
$$+ \deg_H(w) + |E(V(C), w)| \geqslant |E(V(C), V(H))| + 2 + |E(\langle a, u, w \rangle)|.$$

Proof. Define $\varepsilon_0 = |E(u, w)|$, $\varepsilon_1 = |E(a, u)|$ and $\varepsilon_2 = |E(a, w)|$. If we replace C by a claw contained in $\langle a, b_p, u, w \rangle$ and set $H' = \langle (V(H) - \{u, w\}) \cup (V(C) - \{a, b_p\}) \rangle$, then by the maximality of $|E(H)|$,

$$0 \leqslant |E(H)| - |E(H')|$$
$$\leqslant (\deg_H(u) + \deg_H(w) - \varepsilon_0) - |E(B - \{b_p\}, V(H) - \{u, w\})|.$$

Hence

$$|E(B - \{b_p\}, V(H) - \{u, w\})| \leqslant \deg_H(u) + \deg_H(w) - \varepsilon_0.$$

Consequently,

$$\begin{aligned}
&|E(a,V(H))| + |E(b_p,V(H))| + |E(V(C),u)| + |E(V(C),w)| \\
&\quad = |E(V(C),V(H))| + |E(\{a,b_p\},\{u,w\})| \\
&\qquad -|E(V(C)-\{a,b_p\},V(H)-\{u,w\})| \\
&\quad \geqslant |E(V(C),V(H))| + (2+\varepsilon_1+\varepsilon_2) - (\deg_H(u)+\deg_H(w)-\varepsilon_0) \\
&\quad = |E(V(C),V(H))| + 2 + |E(\langle a,u,w\rangle)| - (\deg_H(u)+\deg_H(w)),
\end{aligned}$$

which is equivalent to the desired inequality. □

4. Counting argument

In order to prove the main theorem, we shall choose one, two or three $C^{(i)}$'s, and we shall show that they together with some vertices in H contain more claws, which contradicts the assumption that G is a counterexample. In this section, we shall find a *good* vertex in H that can be used later to find an extra claw. Recall that U is the set of vertices contained in the triangular components of H, and $W = V(H) - U$. We define

$$I = \{i \mid 1 \leqslant i \leqslant k-1,\ E(V(C^{(i)}),W) = \emptyset\},$$

$$J = \{i \mid 1 \leqslant i \leqslant k-1,\ i \notin I,\ |E(V(C^{(i)}),V(H))| > n-s\}.$$

Note that since $n \geqslant 10$ and $s \leqslant 2$, it follows that $|E(V(C^{(i)}),V(H))| \geqslant 9$ if $i \in J$.

Lemma 4.1. *There exists a vertex $v \in W$ such that*

$$\deg_H(v) + \sum_{i \in J} |E(V(C^{(i)}),v)| \geqslant |J| + 3.$$

Proof. We set $l = |I|$ and $m = |J|$, and assume that $\deg_H(v) + \sum_{i \in J} |E(V(C^{(i)}),v)| \leqslant m+2$ for all $v \in W$.

We first claim that $|E(V(C^{(i)}),U)| \leqslant 12$ for each $i \in I$. If $V(C^{(i)})$ is joined by edges to at most one component of $\langle U \rangle$, then the claim is obvious. If $V(C^{(i)})$ is joined to at least two components of $\langle U \rangle$, then by Lemma 3.9, $|E(V(C^{(i)}),U)| \leqslant 3s < 12$. Thus the claim follows. Note that this claim implies that

$$\sum_{i \in I} |E(V(C^{(i)}),U)| \leqslant 12l. \tag{4.1}$$

For $i \in J$, since $E(V(C^{(i)}),W) \neq \emptyset$, it follows from Lemma 3.9 that $|E(V(C^{(i)}),U)| \leqslant 3s$ holds. Hence

$$\sum_{i \in J} |E(V(C^{(i)}),U)| \leqslant 3sm. \tag{4.2}$$

Also, by the definition of I and J, if $i \notin I \cup J$, then

$$|E(V(C^{(i)}), V(H))| \leqslant n - s. \tag{4.3}$$

Now, we shall estimate the following weighted sum of the degrees of the vertices in H in two ways: $\frac{2}{3}\sum_{u \in U} \deg_G(u) + \sum_{v \in W} \deg_G(v)$. First, since $\delta(G) \geqslant k+2$, we have

$$\frac{2}{3}\sum_{u \in U} \deg_G(u) + \sum_{v \in W} \deg_G(v) \geqslant (k+2)\left(\frac{2}{3}|U| + |W|\right) = (k+2)(n-s). \tag{4.4}$$

On the other hand, using inequalities (4.1), (4.2) and (4.3), we have

$$\begin{aligned}
&\frac{2}{3}\sum_{u \in U} \deg_G(u) + \sum_{v \in W} \deg_G(v) \\
&= \frac{2}{3}\sum_{u \in U}\left(\deg_H(u) + \sum_{i=1}^{k-1} |E(V(C^{(i)}), u)|\right) \\
&\quad + \sum_{v \in W}\left(\deg_H(v) + \sum_{i=1}^{k-1} |E(V(C^{(i)}), v)|\right) \\
&= \frac{2}{3}\left(\sum_{u \in U} \deg_H(u) + \left(\sum_{i \in I} + \sum_{i \in J} + \sum_{i \notin I \cup J}\right) |E(V(C^{(i)}), U)|\right) \\
&\quad + \sum_{v \in W}\left(\deg_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)|\right) + \sum_{i \notin I \cup J} |E(V(C^{(i)}), W)| \\
&\leqslant \frac{2}{3}\left(\sum_{u \in U} \deg_H(u) + \sum_{i \in I} |E(V(C^{(i)}), U)| + \sum_{i \in J} |E(V(C^{(i)}), U)|\right) \\
&\quad + \sum_{v \in W}\left(\deg_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)|\right) + \sum_{i \notin I \cup J} |E(V(C^{(i)}), V(H))| \\
&\leqslant \frac{2}{3}(6s + 12l + 3sm) + (m+2)(n-3s) + (n-s)(k-1-l-m) \\
&= (k+2)(n-s) + 8l - (l+1)(n-s) \\
&\leqslant (k+2)(n-s) - 8.
\end{aligned}$$

This contradicts inequality (4.4). □

In the following argument, we consider the vertices in W satisfying the condition in Lemma 4.1. We define

$$W_0 = \left\{ v \in W \mid \deg_H(v) + \sum_{i \in J} |E(V(C^{(i)}), v)| \geqslant |J| + 3 \right\},$$

which is not empty by Lemma 4.1. We also define

$$W_1 = \{v \in W \mid \exists i \in J,\ \deg_H(v) + |E(V(C^{(i)}), v)| \geqslant 4\},$$

$$W_2 = \left\{ v \in W - W_1 \mid \exists J_0 \subset J, |J_0| = 2, \deg_H(v) + \sum_{i \in J_0} |E(V(C^{(i)}), v)| \geqslant 5 \right\},$$

$$W_3 = \left\{ v \in W - W_1 - W_2 \mid \exists J_0 \subset J, |J_0| = 3, \deg_H(v) + \sum_{i \in J_0} |E(V(C^{(i)}), v)| \geqslant 6 \right\}.$$

Lemma 4.2. *The following statements hold:*
(i) $W_0 \subset W_1 \cup W_2 \cup W_3$.
(ii) *If v is a vertex in W_0 with $\deg_H(v) = 1$, then $v \in W_1 \cup W_2$.*
(iii) *If v is a vertex in W_0 with $\deg_H(v) = 2$, then $v \in W_1$.*

Proof. Suppose that $v \in W_0$. By the definition of W_0,

$$\sum_{i \in J} (|E(V(C^{(i)}), v)| - 1) \geqslant 3 - \deg_H(v). \tag{4.5}$$

If $\deg_H(v) = 2$, then by (4.5), there exists $i \in J$ such that $|E(V(C^{(i)}), v)| - 1 \geqslant 1$, implying that $v \in W_1$. If $\deg_H(v) = 1$, then by (4.5), either there exists $i \in J$ such that $|E(V(C^{(i)}), v)| - 1 \geqslant 2$, which implies that $v \in W_1$, or there exist $i, j \in J$ with $i \neq j$ such that $(|E(V(C^{(i)}), v)| - 1) + (|E(V(C^{(j)}), v)| - 1) \geqslant 2$, which implies that $v \in W_2$. Also, if $\deg_H(v) = 0$, then by (4.5), there exists $J_0 \subset J$ with $|J_0| \leqslant 3$ such that $\sum_{i \in J_0} (|E(V(C^{(i)}), v)| - 1) \geqslant 3$, which implies that $v \in W_1 \cup W_2 \cup W_3$. □

Lemma 4.3. *If $v \in W_1$, then $\deg_H(v) = 1$ or 2.*

Proof. Suppose that $C = C^{(i)}$ with $i \in J$ is the one satisfying $\deg_H(v) + |E(V(C), v)| \geqslant 4$. If we assume that $\deg_H(v) = 0$, then we have $|E(V(C), v)| = 4$, and hence v is adjacent to all vertices in C. In particular, since $av \in E(G)$, we see from Lemma 3.1 that

$$|E(b_p, V(H) - \{v\})| \leqslant \deg_H(v) = 0 \quad \text{for all } b_p \in B.$$

Also, since $b_1 v$, $b_2 v$, $b_3 v \in E(G)$, we see from Lemma 3.2 that

$$|E(a, V(H) - \{v\})| \leqslant \deg_H(v) = 0.$$

It follows that $|E(V(C), V(H))| = |E(V(C), v)| = 4$. On the other hand, since $i \in J$, we have $|E(V(C), V(H))| > n - s \geqslant 8$. This is a contradiction. □

5. Proof of the main theorem

In this section, we continue with the notation of the preceding sections, and complete the proof of the main theorem. We first consider the case where $W_1 = \emptyset$.

Case 1: $W_1=\emptyset$.

We take a vertex $v\in W_0$ and fix it. By Lemma 4.2(i), $v\in W_2\cup W_3$. Also by Lemma 4.2(iii), $\deg_H(v)\leqslant 1$. Let J_0 be a subset of J satisfying the conditions in the definition of W_2 or W_3, namely, $2\leqslant|J_0|\leqslant 3$ and

$$\deg_H(v)+\sum_{i\in J_0}|E(V(C^{(i)}),v)|\geqslant|J_0|+3. \tag{5.1}$$

By the definition of W_2 and W_3, for each $i\in J_0$, we have $|E(V(C^{(i)}),v)|\geqslant 2$.

Lemma 5.1. *For each $C=C^{(i)}$ with $i\in J_0$, one of the following statements holds:*
(i) $|E(a,V(H)-\{v\}-N_H(v))|\geqslant 3-\deg_H(v)$.
(ii) $av\notin E(G)$ *and* $|E(b_p,V(H)-\{v\}-N_H(v))|\geqslant 2|J_0|$ *for some* $b_p\in B$.

Proof. Since $i\in J_0\subset J$, we have

$$|E(V(C),V(H))|\geqslant n-s+1\geqslant 9. \tag{5.2}$$

Since $v\notin W_1$ and $i\in J_0$, $2\leqslant|E(V(C),v)|\leqslant 3$. We assume that statement (i) does not hold. Then we have

$$|E(a,V(H)-\{v\})|\leqslant(2-\deg_H(v))+|N_H(v)|=2.$$

If $av\in E(G)$, then by Lemma 3.1, $|E(B,V(H)-\{v\})|\leqslant 3\deg_H(v)\leqslant 3$, and hence $|E(V(C),V(H))|=|E(V(C),v)|+|E(a,V(H)-\{v\})|+|E(B,V(H)-\{v\})|\leqslant 3+2+3=8$, a contradiction to (5.2). This shows that $av\notin E(G)$. Therefore

$$|E(a,V(H))|=|E(a,V(H)-\{v\})|\leqslant 2. \tag{5.3}$$

By (5.2) and (5.3), we have $|E(B,V(H))|\geqslant 7$. This implies that there exists a vertex $b_p\in B$ with $|E(b_p,V(H))|\geqslant 3$. We fix such a vertex b_p.

Let u and w be any two vertices in $N(b_p)\cap V(H)$. If $u\in W$, then since $W_1=\emptyset$, we have

$$\deg_H(u)+|E(V(C),u)|\leqslant 3.$$

If $u\in U$, then by Lemma 3.10, we have $|E(V(C),u)|\leqslant 1$. Hence the above inequality holds even if $u\in U$. Similarly, we have $\deg_H(w)+|E(V(C),w)|\leqslant 3$. Then, by Lemma 3.12,

$$\begin{aligned}|E(b_p,V(H))| &\geqslant |E(V(C),V(H))|+2+|E(\langle a,u,w\rangle)|-|E(a,V(H))|\\ &\quad -(\deg_H(u)+|E(V(C),u)|)-(\deg_H(w)+|E(V(C),w)|)\\ &\geqslant 9+2+|E(\langle a,u,w\rangle)|-|E(a,V(H))|-3-3.\\ &=5+|E(\langle a,u,w\rangle)|-|E(a,V(H))|.\end{aligned} \tag{5.4}$$

Suppose that $E(a,V(H))\neq\emptyset$. Then by Lemma 3.4,

$$|E(b_p,V(H))|=3,\quad\text{and } N(a)\cap V(H)\subset N(b_p)\cap V(H).$$

We may assume that u or w was chosen from $N(a)\cap V(H)$. Then $E(\langle a,u,w\rangle)\neq\emptyset$, and hence by (5.4), $3=|E(b_p,V(H))|\geqslant 6-|E(a,V(H))|$. This contradicts (5.3).

Thus we have $E(a,V(H))=\emptyset$. Then by (5.2), $|E(B,V(H))|\geqslant 9$. On the other hand, by (5.4), $|E(b_p,V(H))|\geqslant 5$. If $|E(b_p,V(H))|\leqslant 6$, then there exists a vertex $b_q\in B$ with $b_q\neq b_p$ such that $|E(b_q,V(H))|\geqslant 2$. This contradicts Lemma 3.5. Thus we have $|E(b_p,V(H))|\geqslant 7$, and hence

$$|E(b_p,V(H)-\{v\}-N_H(v))|\geqslant 6-\deg_H(v).$$

Since $6-\deg_H(v)\geqslant 4$, statement (ii) immediately follows if $|J_0|=2$. If $|J_0|=3$, or equivalently if $v\in W_3$, then by Lemma 4.2(ii), we have $\deg_H(v)=0$, and hence

$$|E(b_p,V(H)-\{v\}-N_H(v))|\geqslant 6=2|J_0|.$$

This completes the proof of Lemma 5.1. □

Now, we shall find $|J_0|+1$ vertex-disjoint claws in $\langle\bigcup_{i\in J_0}V(C^{(i)})\cup V(H)\rangle$, contradicting the assumption that G is a counterexample. We may assume that $J_0=\{i\mid 1\leqslant i\leqslant |J_0|\}$. We may also assume that for an integer h with $0\leqslant h\leqslant |J_0|$, $C=C^{(i)}$ satisfies (i) in Lemma 5.1 for all $1\leqslant i\leqslant h$, and $C=C^{(i)}$ satisfies (ii) in Lemma 5.1 for all $h+1\leqslant i\leqslant |J_0|$. Moreover, for $C=C^{(i)}$ with $h+1\leqslant i\leqslant |J_0|$, we can assume that b_1 is the vertex b_p satisfying the condition of Lemma 5.1(ii).

By the assumption that $v\in W_2\cup W_3$, we have (5.1), or equivalently,

$$\sum_{i\in J_0}(|E(V(C^{(i)}),v)|-1)\geqslant 3-\deg_H(v).$$

This inequality implies that for each i $(1\leqslant i\leqslant |J_0|)$, we can choose a subset $X^{(i)}\subset N(v)\cap V(C^{(i)})$ such that

$$|X^{(i)}|\leqslant |E(V(C^{(i)}),v)|-1,$$

$$\sum_{i=1}^{|J_0|}|X^{(i)}|=3-\deg_H(v)$$

and

$$a^{(i)}\notin X^{(i)}\quad\text{for } 1\leqslant i\leqslant h,$$

$$b_1^{(i)}\notin X^{(i)}\quad\text{for } h+1\leqslant i\leqslant |J_0|.$$

Then we can find a claw with center v in $\langle\{v\}\cup N_H(v)\cup\bigcup_{i=1}^{|J_0|}X^{(i)}\rangle$. Note that by the condition in Lemma 5.1(ii), we have $a^{(i)}\notin X^{(i)}$ also for $h+1\leqslant i\leqslant |J_0|$.

We define $Y^{(i)}=V(C^{(i)})-X^{(i)}$ for $1\leqslant i\leqslant h$ and $Y^{(i)}=\{a^{(i)},b_1^{(i)}\}$ for $h+1\leqslant i\leqslant |J_0|$. We take disjoint subsets $Z^{(i)}$ of $V(H)-\{v\}-N_H(v)$ for $1\leqslant i\leqslant |J_0|$ such that

$$Z^{(i)}\subset N(a^{(i)}),\ |Z^{(i)}|=|X^{(i)}|\quad\text{for } 1\leqslant i\leqslant h,$$

$$Z^{(i)}\subset N(b_1^{(i)}),\ |Z^{(i)}|=2\quad\text{for } h+1\leqslant i\leqslant |J_0|.$$

This can be done by determining $Z^{(i)}$ from $i=1$ up to $|J_0|$, because for $1 \leqslant i \leqslant h$,

$$\sum_{j=1}^{i} |X^{(j)}| \leqslant \sum_{j=1}^{|J_0|} |X^{(j)}| = 3 - \deg_H(v) \leqslant |E(a^{(i)}, V(H) - \{v\} - N_H(v))|,$$

and for $h+1 \leqslant i \leqslant |J_0|$,

$$\sum_{j=1}^{h} |X^{(j)}| + 2(i-h) \leqslant 2i \leqslant 2|J_0| \leqslant |E(b_1^{(i)}, V(H) - \{v\} - N_H(v))|.$$

Then for each i with $1 \leqslant i \leqslant |J_0|$, $\langle Y^{(i)} \cup Z^{(i)} \rangle$ contains a claw with center $a^{(i)}$ or $b_1^{(i)}$ depending on whether $i \leqslant h$ or $i \geqslant h+1$. Obviously, these claws and the claw in $\langle \{v\} \cup N_H(v) \cup \bigcup_{i=1}^{|J_0|} X^{(i)} \rangle$ are pairwise vertex-disjoint. This contradicts the assumption that G is a counterexample, and completes the proof for Case 1.

Case 2: $W_1 \neq \emptyset$.

Let $v \in W_1$. By Lemma 4.3, $\deg_H(v) = 1$ or 2. If there exists a vertex $v \in W_1$ with $\deg_H(v) = 2$, then we take such a vertex v. By the definition of W_1, we can take a claw $C = C^{(i)}$ with $i \in J$ such that $\deg_H(v) + |E(V(C), v)| \geqslant 4$. Then by Lemma 3.11, we have $E(V(C), U) = \emptyset$, and hence

$$|E(V(C), W)| = |E(V(C), V(H))| \geqslant n - s + 1 \geqslant 9. \tag{5.5}$$

We distinguish several cases depending on the degree of v in H and the degrees of the neighbors of v in H.

Case 2.1: $\deg_H(v) = 2$.

Let $N_H(v) = \{u, w\}$. We may assume that v and C are chosen so that $\deg_H(u) + \deg_H(w)$ is maximum, and subject to this condition, $|E(B, v)|$ is maximum. Note that since $\deg_H(v) + |E(V(C), v)| \geqslant 4$, we have

$$|E(V(C), v)| \geqslant 2. \tag{5.6}$$

Hence we have $E(B, v) \neq \emptyset$. By Lemma 3.3, we have

$$N(a) \cap V(H) \subset \{u, v, w\}. \tag{5.7}$$

Subcase 2.1.1: $\deg_H(u) = \deg_H(w) = 2$.

Lemma 5.2. *The following statements hold*:

(i) $|E(B, W - \{u, v, w\})| \leqslant 4$.

(ii) *If* $|E(B, v)| \geqslant 2$, *then* $|E(b_p, W - \{u, v, w\})| \leqslant 1$ *for any* $b_p \in B$, *and hence* $|E(B, W - \{u, v, w\})| \leqslant 3$.

(iii) $|E(B, W - \{u, w\})| \leqslant 6$. *The equality* $|E(B, W - \{u, w\})| = 6$ *holds only if* $|E(B, v)| = 3$.

Proof. The statement (iii) is immediate from (i) and (ii). So we shall prove (i) and (ii) only.

Let $b_q \in N(v) \cap B$ and $b_p \in B$ with $b_p \neq b_q$. Suppose that $|E(b_p, W - \{u,v,w\})| \geqslant 2$. Then, $\langle b_q, u, v, w \rangle$ contains a claw with center v, and $\langle \{a, b_p\} \cup (N(b_p) \cap (W - \{u,v,w\})) \rangle$ contains a claw with center b_p, a contradiction. Hence it follows that $|E(b_p, W - \{u,v,w\})| \leqslant 1$ for any $b_p \in B - \{b_q\}$. Since the vertex $b_q \in N(v) \cap B$ was also arbitrary, if $|N(v) \cap B| = |E(B,v)| \geqslant 2$, then $|E(b_p, W - \{u,v,w\})| \leqslant 1$ holds for any $b_p \in B$. This shows (ii).

To show (i), we suppose that $|E(B,v)| = 1$ and set $N(v) \cap B = \{b_q\}$. The argument in the preceding paragraph shows that $|E(B - \{b_q\}, W - \{u,v,w\})| \leqslant 2$. On the other hand, by (5.6), v is adjacent to a. Hence by Lemma 3.1, $|E(b_q, W - \{v\})| \leqslant \deg_H(v) = 2$. Thus $|E(B, W - \{u,v,w\})| \leqslant |E(B - \{b_q\}, W - \{u,v,w\})| + |E(b_q, W - \{v\})| \leqslant 4$, showing (i). □

Lemma 5.3. $|E(V(C), \{u,w\})| \geqslant 3$.

Proof. Suppose that $|E(V(C), \{u,w\})| \leqslant 2$. By Lemma 5.2(iii), we have $|E(B, W - \{u,w\})| \leqslant 6$. Also by (5.7), we have $E(a, W - \{u,w\}) \subset \{av\}$. Hence by (5.5),

$$9 \leqslant n - s + 1 \leqslant |E(V(C), W)| \tag{5.8}$$
$$= |E(B, W - \{u,w\})| + |E(a, W - \{u,w\})| + |E(V(C), \{u,w\})|$$
$$\leqslant 6 + 1 + 2 = 9. \tag{5.9}$$

Thus equality holds in (5.8) and (5.9). The equality $|E(a, W - \{u,w\})| = 1$ implies that $av \in E(G)$. The equality $|E(B, W - \{u,w\})| = 6$ implies, by Lemma 5.2(iii), that $|E(B,v)| = 3$. Hence $N(v) \supset V(C)$. Also, from the equality $9 = n - s + 1$, we have $n = 10$ and $s = 2$, and consequently, $|W| = n - 3s = 4$. Let $W = \{u,v,w,x\}$. Then, since $\deg_H(u) = \deg_H(v) = \deg_H(w) = 2$ and $\langle W \rangle$ does not have a triangular component, we obtain $\deg_H(x) = 2$. Now, applying Lemma 3.6 to v and x, we have $|E(V(C), x)| \leqslant 1$. Consequently,

$$|E(V(C), W)| = |E(V(C), v)| + |E(V(C), \{u,w\})| + |E(V(C), x)| \leqslant 4 + 2 + 1 = 7,$$

which contradicts (5.5). □

Lemma 5.4. $au, aw \notin E(G)$.

Proof. Assume the contrary, say $au \in E(G)$. Since $w \notin N_H(u)$, applying Lemma 3.3 to the vertex w, we obtain $E(B,w) = \emptyset$.

We first suppose that $aw \in E(G)$. Then by the same argument, we have $E(B,u) = \emptyset$. It follows that $E(V(C), \{u,w\}) \subset \{au, aw\}$, which contradicts the result of Lemma 5.3.

Thus $aw \notin E(G)$. By Lemma 5.3, $|E(V(C), u)| = |E(V(C), \{u,w\})| \geqslant 3$. Then, in view of Lemma 3.6 and (5.6), we obtain $N(u) \cup N(v) \not\supset V(C)$. In particular, we have $|E(V(C), u)| = |E(V(C), \{u,w\})| = 3$. Since $au \in E(G)$, v is not adjacent to all vertices of

B. This means that the equality in Lemma 5.2(iii) does not hold, i.e., $|E(B, W - \{u,w\})| \leqslant 5$. Thus we have

$$\begin{aligned} 9 \leqslant n-s+1 &\leqslant |E(V(C), W)| \qquad (5.10)\\ &= |E(B, W-\{u,w\})| + |E(a, W-\{u,w\})| + |E(V(C), \{u,w\})|\\ &\leqslant 5+1+3=9. \qquad (5.11)\end{aligned}$$

Hence the equality holds in (5.10) and (5.11). From the equality in (5.10), we have $n=10$, $s=2$ and $|W|=4$. Let $W=\{u,v,w,x\}$. Then $\deg_H(x)=2$. From the equality in (5.11), it follows that $av \in E(G)$ and

$$5 = |E(B, W-\{u,w\})| = |E(B, \{v,x\})|.$$

This implies that $N(v) \cup N(x) \supset V(C)$, $|E(V(C), v)| \geqslant 2$ and $|E(V(C),x)| \geqslant 2$. This contradicts Lemma 3.6. We complete the proof of Lemma 5.4. □

Lemma 5.5. $av \notin E(G)$.

Proof. Suppose that $av \in E(G)$. By Lemma 5.4 and (5.7), we have $E(a, W)=\{av\}$. By Lemma 3.1, for each vertex $b_p \in B$, we have $|E(b_p, W-\{v\})| \leqslant \deg_H(v)=2$, and hence $|E(B, W-\{v\})| \leqslant 6$. By (5.5),

$$\begin{aligned} 9 \leqslant |E(V(C), W)| &= |E(B, W-\{v\})| + |E(B,v)| + |E(a, W)|\\ &\leqslant 6 + |E(B,v)| + 1. \qquad (5.12)\end{aligned}$$

Hence we have $|E(B,v)| \geqslant 2$. We may assume that $b_1, b_2 \in N(v)$.

On the other hand, by Lemmas 5.3 and 5.4,

$$3 \leqslant |E(V(C), \{u,w\})| = |E(B, \{u,w\})|.$$

Hence we may assume that $|E(B,u)| \geqslant 2$. Now, in view of Lemma 3.6, it follows that $N(u) \cup N(v) \not\supset V(C)$. This implies that

$$N(v) \cap V(C) = \{a, b_1, b_2\} \quad \text{and} \quad N(u) \cap B = \{b_1, b_2\}. \qquad (5.13)$$

In particular, $|E(B,v)|=2$, and hence equality holds in (5.12). Thus we have $|W|=4$, and we can set $W=\{u,v,w,x\}$ so that $\deg_H(x)=2$. Moreover, the equality $|E(B, W - \{v\})|=6$ implies that $|E(b_p, W-\{v\})|=2$ holds for each $b_p \in B$. By (5.13), we have $b_3 u \notin E(G)$. Hence $b_3 w$ and $b_3 x \in E(G)$. Therefore, we have $N(v) \cup N(w) \supset V(C)$ and $N(v) \cup N(x) \supset V(C)$. Applying Lemma 3.6 to the vertices v and w, we have $|E(V(C), w)| \leqslant 1$. Similarly, applying Lemma 3.6 to v and x, we have $|E(V(C),x)| \leqslant 1$. Consequently,

$$\begin{aligned} |E(V(C), W)| &= |E(V(C),u)| + |E(V(C),v)| + |E(V(C),w)| + |E(V(C),x)|\\ &\leqslant 2+3+1+1=7,\end{aligned}$$

which contradicts (5.5). This shows that $av \notin E(G)$. □

We are now in a position to complete the discussion for Subcase 2.1.1. By (5.7) and Lemmas 5.4 and 5.5, we have $E(a,W)=\emptyset$. By Lemma 5.3, $|E(B,\{u,w\})|\geqslant 3$. We may assume that $|E(B,u)|\geqslant 2$, say $b_1,b_2\in N(u)$. Now from (5.6), we have $|E(B,v)|=|E(V(C),v)|\geqslant 2$.

First suppose that $|E(B,v)|=2$. By Lemma 3.7(i), we have $|E(B,\{u,w\})|\leqslant 4$, and by Lemma 5.2(ii), we have $|E(B,W-\{u,v,w\})|\leqslant 3$. Hence we obtain

$$\begin{aligned} 9 &\leqslant |E(V(C),W)|=|E(B,W)| & (5.14)\\ &= |E(B,v)|+|E(B,\{u,w\})|+|E(B,W-\{u,v,w\})| \\ &\leqslant 2+4+3=9, & (5.15)\end{aligned}$$

in which equality must hold. The equality in (5.14) implies that $|W|=4$, and hence we can set $W=\{u,v,w,x\}$ so that $\deg_H(x)=2$. Applying Lemma 3.7(i) to the vertices u and $N_H(u)=\{v,x\}$, we have $|E(B,\{v,x\})|\leqslant 4$, while the equality in (5.15) implies that $|E(B,\{v,x\})|=5$. This is a contradiction.

Next suppose that $|E(B,v)|=3$. We claim that $E(b_1,W-\{u,v,w\})=\emptyset$. If $b_1z\in E(G)$ for some vertex $z\in W-\{u,v,w\}$, then $\langle a,b_1,u,z\rangle$ contains a claw with center b_1 and $\langle b_2,b_3,v,w\rangle$ contains a claw with center v, a contradiction. Hence $E(b_1,W-\{u,v,w\})=\emptyset$, as claimed. Similarly, we have $E(b_2,W-\{u,v,w\})=\emptyset$. By Lemma 5.2, we have $|E(b_3,W-\{u,v,w\})|\leqslant 1$, and hence $|E(B,W-\{u,v,w\})|\leqslant 1$. Since $|E(B,\{u,w\})|\leqslant 4$ by Lemma 3.7(i), we obtain

$$\begin{aligned} 9 &\leqslant |E(V(C),W)|=|E(B,W)| \\ &= |E(B,v)|+|E(B,\{u,w\})|+|E(B,W-\{u,v,w\})| \\ &\leqslant 3+4+1=8,\end{aligned}$$

which is a contradiction.

This completes the proof of the main theorem in Subcase 2.1.1.

Subcase 2.1.2: $\deg_H(u)+\deg_H(w)=3$.

We may assume that $\deg_H(u)=1$ and $\deg_H(w)=2$. We may also assume that $N(v)\cap B=\{b_p\mid 1\leqslant p\leqslant |E(B,v)|\}$. First we prove the following lemma, which concerns the cardinality of $E(B,W-\{u,v,w\})$ in connection with $|E(B,v)|$.

Lemma 5.6. (i) *If* $|E(B,v)|=3$, *then* $|E(B,W-\{u,v,w\})|\leqslant 1$.

(ii) *If* $|E(B,v)|=2$, *then* $|E(\{b_1,b_3\},W-\{u,v,w\})|\leqslant 1$ *and* $|E(\{b_2,b_3\},W-\{u,v,w\})|\leqslant 1$; *consequently*, $|E(B,W-\{u,v,w\})|\leqslant 2$ *holds.*

(iii) *If* $|E(B,v)|=1$, *then* $|E(\{b_2,b_3\},W-\{u,v,w\})|\leqslant 1$ *and* $|E(b_1,W-\{u,v,w\})|\leqslant 2$; *consequently*, $|E(B,W-\{u,v,w\})|\leqslant 3$ *holds.*

(iv) *In any case*, $|E(B,W-\{u,w\})|\leqslant 4$ *holds.*

Proof. In general, if $b_pv\in E(G)$, then by replacing C by a claw in $\langle b_p,u,v,w\rangle$, we see from the maximality of $|E(H)|$ that

$$|E(B-\{b_p\},W-\{u,v,w\})|\leqslant 1.$$

This shows (i), (ii) and the first inequality in (iii). The second inequality in (iii) follows from Lemma 3.1, since (5.6) and the assumption $|E(B,v)|=1$ implies that $av \in E(G)$. Statement (iv) is immediate from (i), (ii) and (iii). □

Lemma 5.7. $au \notin E(G)$.

Proof. Assume that $au \in E(G)$. Then by Lemma 3.1, $|E(b_p, W - \{u\})| \leqslant 1$ for all $b_p \in B$, and hence $|E(B, W-\{u\})| \leqslant 3$. On the other hand, by (5.7), we have $|E(a, W - \{u\})| \leqslant |E(a, \{v,w\})| \leqslant 2$. Hence

$$\begin{aligned}9 &\leqslant |E(V(C),W)| = |E(B, W - \{u\})| + |E(a, W - \{u\})| + |E(V(C),u)| \\ &\leqslant 3+2+4=9. \qquad (5.16)\end{aligned}$$

Thus equality holds in (5.16). In particular, we have $|E(V(C),u)|=4$, and hence u is adjacent to all vertices of B. By Lemma 3.2, we obtain $|E(a, W-\{u\})| \leqslant \deg_H(u) = 1$, while the equality in (5.16) implies that $|E(a, W - \{u\})| = 2$. This is a contradiction. □

We write $N_H(w)=\{v,x\}$.

Lemma 5.8. $\deg_H(x)=1$.

Proof. Assume that $\deg_H(x)=2$. Then, W contains at least 5 vertices implying that $|E(V(C),W)| \geqslant 10$.

On the other hand, since $v \in W_1$ was chosen so that $\deg_H(u)+\deg_H(w)$ is maximum, w is not contained in W_1. This implies that $|E(V(C),w)| \leqslant 1$. Also by (5.7) and Lemma 5.7, we have $E(a, W-\{w\}) \subset \{av\}$, and hence $|E(a, W-\{w\})| \leqslant 1$. Now, using Lemma 5.6(iv), we obtain

$$\begin{aligned}10 &\leqslant |E(V(C),W)| \\ &= |E(B, W - \{u,w\})| + |E(B,u)| + |E(a, W - \{w\})| + |E(V(C),w)| \\ &\leqslant 4+3+1+1=9.\end{aligned}$$

This is a contradiction. □

By Lemma 5.8, $\langle u,v,w,x \rangle$ is a component of H isomorphic to the path of length three. Hence by Lemma 3.8(i), we have $|E(B,\{u,w\})| \leqslant 4$. Also by Lemma 5.6(iv), we have $|E(B, W-\{u,w\})| \leqslant 4$. Since $N(a) \cap W \subset \{v,w\}$ by (5.7) and Lemma 5.7, we obtain

$$\begin{aligned}9 &\leqslant |E(V(C),W)| = |E(B,\{u,w\})| + |E(B, W - \{u,w\})| + |E(a,W)| \\ &\leqslant 4+4+2=10.\end{aligned}$$

This implies that

$$3 \leqslant |E(B,\{u,w\})| \leqslant 4, \tag{5.17}$$

$$3 \leqslant |E(B,W-\{u,w\})| \leqslant 4, \tag{5.18}$$

and

$$N(a) \cap \{v,w\} \neq \emptyset. \tag{5.19}$$

Lemma 5.9. $|E(B,v)| \leqslant 2$.

Proof. Assume that $|E(B,v)| = 3$. Then by (5.19), $N(v) \cup N(w) \supset V(C)$. Hence by Lemma 3.6, we have $|E(V(C),w)| \leqslant 1$. Consequently, we obtain

$$\begin{aligned} 9 &\leqslant |E(V(C),W)| \\ &= |E(V(C),w)| + |E(B,W-\{u,w\})| + |E(B,u)| + |E(a,W-\{w\})| \\ &\leqslant 1+4+3+1 = 9, \end{aligned}$$

in which equality holds. In particular, we have $|E(B,u)| = 3$ and $|E(V(C),w)| = 1$. Let $z \in V(C)$ be the neighbor of w. If $z \in B$, then we have $|E(B,\{u,w\})| = 4$ but in Lemma 3.8(ii) neither (a) nor (b) holds, a contradiction. If $z = a$, then $\langle B \cup \{u\} \rangle$ contains a claw with center u, and $\langle a,v,w,x \rangle$ contains a claw with center w. This is a contradiction. Thus the lemma follows. □

Lemma 5.10. $|E(B,\{u,w\})| = 3$.

Proof. Assume the contrary, i.e., by (5.17), $|E(B,\{u,w\})| = 4$. By Lemma 3.8(ii), we obtain $|E(B,w)| \geqslant 2$, and hence $w \in W_1$. By the maximality of $|E(B,v)|$ in our choice of $v \in W_1$, it follows from Lemma 5.9 that $|E(B,v)| = |E(B,w)| = 2$. Thus in Lemma 3.8(ii), the situation (a) holds, i.e., we have $N(u) \cap B = N(w) \cap B$. Also, by (5.19), $a \in N(v) \cup N(w)$. In view of Lemma 3.6, we have $N(v) \cup N(w) \not\supset V(C)$, implying that $N(v) \cap B = N(w) \cap B$. Thus we have

$$N(u) \cap B = N(v) \cap B = N(w) \cap B = \{b_1,b_2\}.$$

Applying Lemma 3.1 to a vertex in $N(a) \cap \{v,w\}$, which is not empty by (5.19), we obtain $E(\{b_1,b_2\}, W-\{u,v,w\}) = \emptyset$. Hence by Lemma 5.6(ii),

$$|E(B,W-\{u,v,w\})| = |E(b_3,W-\{u,v,w\})| \leqslant 1.$$

Consequently, we have

$$\begin{aligned} 9 &\leqslant |E(V(C),W)| = |E(B,W-\{u,v,w\})| + |E(B,\{u,v,w\})| + |E(a,W)| \\ &= |E(b_3,W-\{u,v,w\})| + |E(B,\{u,v,w\})| + |E(a,\{v,w\})| \\ &\leqslant 1+6+2 = 9, \end{aligned}$$

and equality holds. In particular, we have $aw \in E(G)$ and $|E(b_3, W - \{u,v,w\})| = 1$. Now, by replacing C by a claw contained in $\langle b_1, b_2, u, v\rangle$, we get a contradiction to the maximality of $|E(H)|$. This completes the proof of Lemma 5.10. □

By Lemma 5.10 and (5.18), we get

$$9 \leqslant |E(V(C), W)| = |E(B, \{u,w\})| + |E(B, W - \{u,w\})| + |E(a, W)|$$
$$\leqslant 3 + 4 + 2 = 9,$$

in which equality holds. Therefore, $W = \{u, v, w, x\}$ and

$$4 = |E(B, W - \{u,w\})| = |E(B, \{v,x\})|, \tag{5.20}$$

$$av, aw \in E(G). \tag{5.21}$$

If $|E(B,v)| = 1$, then by (5.20), we have $|E(B,x)| = 3$. Then by Lemma 3.2, we obtain $|E(a, \{u,v,w\})| \leqslant 1$, which contradicts (5.21). Thus we have $|E(B,v)| = 2$, and hence $|E(B,x)| = 2$ by (5.20). It follows from Lemma 5.6(ii) that $b_1x, b_2x \in E(G)$.

Applying Lemma 3.1 to the vertex w, which is adjacent to a by (5.21), $|E(b_p, \{u, v, x\})| \leqslant 2$ holds for all $b_p \in B$. In particular, since $|E(b_p, \{v,x\})| = 2$ for $p = 1, 2$, u is not adjacent to b_1 or b_2. Hence,

$$N(u) \cap B \subset \{b_3\}. \tag{5.22}$$

On the other hand, if $b_3w \in E(G)$, then we get $N(v) \cup N(w) \supset V(C)$, which contradicts Lemma 3.6. Hence we have

$$N(w) \cap B \subset \{b_1, b_2\}. \tag{5.23}$$

By Lemma 5.10 and (5.22) and (5.23), equality holds in (5.22) and (5.23). Now, replacing C by a claw contained in $\langle b_1, b_2, w, x\rangle$, we get a contradiction to the maximality of $|E(H)|$.

This completes the proof of the main theorem in Subcase 2.1.2.

Subcase 2.1.3: $\deg_H(u) = \deg_H(w) = 1$.

We first prove the following lemma.

Lemma 5.11. $E(B, W - \{u,v,w\}) \neq \emptyset$.

Proof. We assume that $E(B, W - \{u,v,w\}) = \emptyset$. By (5.5) and (5.7), $|E(V(C), W)| = |E(V(C), \{u,v,w\})| \geqslant 9$.

We first suppose that $au \in E(G)$. By Lemma 3.1, $|E(b_p, \{v,w\})| \leqslant 1$ holds for all $b_p \in B$, and hence $|E(B, \{v,w\})| \leqslant 3$. Hence

$$9 \leqslant |E(V(C), \{u,v,w\})| = |E(B, \{v,w\})| + |E(B,u)| + |E(a, \{u,v,w\})|$$
$$\leqslant 3 + 3 + 3 = 9. \tag{5.24}$$

Thus equality holds in (5.24). This in particular implies that $aw \in E(G)$. Consequently, again by Lemma 3.1, we have $|E(b_p, \{u,v\})| \leqslant 1$ for all $b_p \in B$, and hence $|E(B, \{u,v\})|$

$\leqslant 3$. However, since $E(B,v) \neq \emptyset$ by (5.6), the equality $|E(B,u)|=3$ in (5.24) implies that $|E(B,\{u,v\})| \geqslant 4$. This is a contradiction.

Thus we have $au \notin E(G)$. By symmetry, we also have $aw \notin E(G)$, and hence $E(a,\{u,v,w\}) \subset \{av\}$. Since $|E(V(C),\{u,v,w\})| \geqslant 9$, this implies

$$|E(B,\{u,v,w\})| \geqslant 8, \tag{5.25}$$

i.e., B and $\{u,v,w\}$ are joined completely except possibly one pair.

Suppose that $E(\langle B \rangle) \neq \emptyset$. Without loss of generality, we may assume that $b_1b_2 \in E(G)$. By (5.25), we can find a vertex $x \in \{u,v,w\}$ such that x is adjacent to b_1 and b_2, and that b_3 is adjacent to both of the vertices in $\{u,v,w\}-\{x\}$. Then, by replacing C by a claw contained in $\langle \{a,b_3\} \cup (\{u,v,w\} - \{x\}) \rangle$, we get a contradiction to the maximality of $|E(H)|$.

Thus we have $E(\langle B \rangle)=\emptyset$. By (5.25), some vertex $b_p \in B$ is joined to all vertices of $\{u,v,w\}$. Then, by replacing C by a claw contained in $\langle b_p,u,v,w \rangle$, we get a contradiction to the maximality of $\sum_{j=1}^{k-1} |E(\langle V(C^{(j)}) \rangle)|$. □

Lemma 5.12. *For every $b_p \in N(v) \cap B$, we have $E(B-\{b_p\}, W-\{u,v,w\})=\emptyset$.*

Proof. If $E(b_q, W-\{u,v,w\}) \neq \emptyset$ for some $b_q \in B-\{b_p\}$, then by replacing C by a claw contained in $\langle b_p,u,v,w \rangle$, we get a contradiction to the maximality of $|E(H)|$. □

By Lemma 5.11, there exists a vertex $b_p \in B$ with $E(b_p, W-\{u,v,w\}) \neq \emptyset$. We can assume that $p=1$. Then by Lemma 5.12, v cannot be adjacent to either b_2 or b_3. Since $|E(V(C),v)| \geqslant 2$ by (5.6), we have

$$N(v) \cap V(C) = \{a,b_1\}. \tag{5.26}$$

In particular, we have $av \in E(G)$, and hence by Lemma 3.1,

$$|E(b_1, W-\{v\})| \leqslant 2. \tag{5.27}$$

If $au \in E(G)$, then by Lemma 3.1, we have $|E(b_1, W-\{u\})| \leqslant \deg_H(u)=1$, while $b_1v \in E(G)$ and $E(b_1, W-\{u,v,w\}) \neq \emptyset$. This is a contradiction. Hence we have $au \notin E(G)$. By symmetry, we also have $aw \notin E(G)$. By (5.7), we obtain $N(a) \cap W = \{v\}$. Also, by Lemma 5.12, we have $E(\{b_2,b_3\}, W-\{u,v,w\})=\emptyset$. Consequently, by (5.26) and (5.27),

$$\begin{aligned} 9 &\leqslant |E(V(C),W)| \\ &= |E(V(C),v)| + |E(b_1, W-\{v\})| + |E(\{b_2,b_3\}, W-\{v\})| + |E(a, W-\{v\})| \\ &= |E(V(C),v)| + |E(b_1, W-\{v\})| + |E(\{b_2,b_3\}, \{u,w\})| \\ &\leqslant 2+2+4=8, \end{aligned}$$

a contradiction.

This completes the proof of the main theorem in Case 2.1.

Case 2.2: $\deg_H(v) = 1$.

Let $N_H(v) = \{w\}$. Note that $|E(V(C), v)| \geqslant 3$ since $v \in W_1$.

Lemma 5.13. *For* $\{p, q, r\} = \{1, 2, 3\}$, *we suppose that* $b_p, b_q \in N(v) \cap B$. *Then,*

(i) $|E(\{a, b_r\}, W - \{v, w\})| \leqslant 1$, *and*

(ii) *if* $\deg_H(w) = 1$, *then* $E(\{a, b_r\}, W - \{v, w\}) = \emptyset$.

Proof. By replacing C by a claw contained in $\langle b_p, b_q, v, w \rangle$, we obtain both statements immediately from the maximality of $|E(H)|$. □

Lemma 5.14. $|E(B, v)| = 2$.

Proof. Since $|E(V(C), v)| \geqslant 3$, we have $|E(B, v)| \geqslant 2$. Suppose that $|E(B, v)| = 3$. If $\deg_H(w) = 1$, then by Lemma 5.13(ii), no vertex in $V(C)$ is joined to $W - \{v, w\}$. Hence $|E(V(C), W)| = |E(V(C), \{v, w\})| \leqslant 8$, which contradicts (5.5). If $\deg_H(w) = 2$, then $w \notin W_1$ by the choice of $v \in W_1$, and hence $|E(V(C), w)| \leqslant 1$. By Lemma 5.13(i), $|E(\{a, b_r\}, W - \{v, w\})| \leqslant 1$ for each $b_r \in B$, and hence $|E(V(C), W - \{v, w\})| \leqslant 3$. Consequently,

$$\begin{aligned} |E(V(C), W)| &\leqslant |E(V(C), W - \{v, w\})| + |E(V(C), v)| + |E(V(C), w)| \\ &\leqslant 3 + 4 + 1 = 8, \end{aligned}$$

which contradicts (5.5). This proves $|E(B, v)| = 2$. □

Since $|E(V(C), v)| \geqslant 3$, we have $av \in E(G)$ by Lemma 5.14. By Lemma 3.1, we have $|E(b_p, W - \{v\})| \leqslant 1$ for all $b_p \in B$, and hence $|E(B, W - \{v\})| \leqslant 3$. Also, by Lemma 5.13(i), we obtain $|E(a, W - \{v, w\})| \leqslant 1$, implying $|E(a, W - \{v\})| \leqslant 2$. Now, since $|E(V(C), v)| = 3$, we obtain

$$\begin{aligned} |E(V(C), W)| &= |E(V(C), v)| + |E(B, W - \{v\})| + |E(a, W - \{v\})| \\ &\leqslant 3 + 3 + 2 = 8, \end{aligned}$$

which contradicts (5.5).

This completes the proof of the main theorem. □

Note. The referees have pointed out that some weaker asymptotic results can be obtained by other methods, for example, probabilistic methods as in [9,6], or Szemerédi's Regularity Lemma [10]. Also in [8], it is proved by a much simpler argument that a graph of order at least $(t + 1)k + O(t^2)$ with minimum degree $k + t - 1$ contains k pairwise vertex-disjoint stars of order $t + 1$.

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